

Communication-Free Multi-Agent Control under Local Temporal Tasks and Relative-Distance Constraints

Meng Guo
mengg@kth.se

Jana Tumova
tumova@kth.se

Dimos V. Dimarogonas
dimos@kth.se

ACCESS Linnaeus Center
Center for Autonomous Systems
Automatic Control Laboratory
KTH - Royal Institute of Technology
Osquidas väg 10, SE-100 44, Stockholm, Sweden

ABSTRACT

We propose a distributed control and coordination strategy for multi-agent systems where each agent has a local task specified as a Linear Temporal Logic (LTL) formula and at the same time is subject to relative-distance constraints with its neighboring agents. The local tasks capture the temporal requirements on individual agents' behaviors, while the relative-distance constraints impose requirements on the collective motion of the whole team. The proposed solution relies only on relative-state measurements among the neighboring agents without the need for explicit information exchange. It is guaranteed that the local tasks given as syntactically co-safe or general LTL formulas are fulfilled and the relative-distance constraints are satisfied at all time. The approach is demonstrated with computer simulations.

Keywords

Hybrid Systems; Multi-Agent Systems; Autonomous Agents; Formal Methods in Control; Potential Fields; LTL.

1. INTRODUCTION

Cooperative control of multi-agent systems generally focuses on designing local control laws to achieve a global goal, such as reference-tracking [9], consensus [18], or formation [11]. In addition to these objectives, various relative-motion constraints are often imposed to achieve stability, safety and integrity of the overall system, such as collision avoidance [3], network connectivity [11, 23], or relative velocity constraint [9]. This work is motivated by the desire to specify and achieve more structured and complex team behaviors than the listed ones. Particularly, following a recent trend, we consider Linear Temporal Logic (LTL) formulas as suitable descriptions of desired high-level goals. LTL allows to rigorously specify various temporal tasks, including periodic surveillance, sequencing, request-response, and their combinations. Furthermore, with the use of formal verification-inspired methods, a discrete plan that guarantees the specification satisfaction can be automatically synthesized, while various abstraction techniques bridge the continuous control problem and the discrete planning one. As a result, a generic hierarchical approach that allows for correct-by-design control with respect to the given LTL specification has been formulated and largely employed during the last decade or so in single-agent as well as multi-agent settings [2, 4, 7, 13, 15, 16, 20, 21, 22].

In temporal logic-based multi-agent control, two different points of view can be taken: a top-down and a bottom-up. In the former one, a global specification captures requirements on the overall team behavior. Typically, the focus of decentralization is on decomposing the specification into tasks to be executed by the individual agents in a synchronized [2] or partially synchronized [13, 22] manner. A central monitoring unit then ensures that the composition of the local plans yields the satisfaction of the global goal.

In contrast, in bottom-up approach, each agent is assigned its own local task. The tasks can be independent [4, 7] or partially dependent, involving requests for collaboration with the others [21]. A major research interest here is the decentralization of planning and control procedures. For instance, in [7], a decentralized revision scheme is suggested for a team in a partially-known workspace. In [4], gradual verification is employed to ensure that independent LTL tasks remain mutually satisfiable while avoiding collisions. In [21], a receding horizon approach is employed to achieve partially decentralized planning for collaborative tasks. In [20], the authors propose a compositional motion planning framework for multi-robot systems under safe LTL specifications.

In this work, we tackle the multi-agent control problem under local LTL tasks from the bottom-up perspective. Even though the local tasks are mutually independent, the agents within a multi-agent group are often more than a collection of stand-alone systems. Instead, they are subject to dynamic constraints with their neighboring agents and in such a case, integration of the continuous motion control with the high-level discrete network structure control is essential [9, 23]. Here, the agents are subject to relative-distance constraints that need to be satisfied at all times. This coupling constraints make the team of agents competitive as each agent has to satisfy its local task and at the same time cooperative as they have to maintain the relative distance within the team. We addressed a version of this problem in [8], where we proposed a dynamic leader-follower coordination and control scheme as a solution. In this work, however, we aim for a fully decentralized and communication-free solution that is applicable, e.g., to low-cost robotic systems equipped with range and angle sensors, but without communication units. Our solution consists of three ingredients: a standard discrete plan synthesis algorithm, a decentralized, hybrid, potential-field-based motion controller with two different control modes and a switching strategy between the two different continuous control modes.

In summary, we propose a fully communication-free decentralized hybrid control scheme for multi-agent systems under both complex high-level local LTL tasks and low-level relative-distance constraints. Specifically, our main contribution is the design of a two-mode communication-free control law that brings a group of agents to a region of interest.

The rest of the paper is organized as follows. Sec. 2 introduces preliminaries. Sec. 3 formalizes the considered problem. Sec. 4 presents our solution in details. Sec. 5 demonstrates the results in simulations. We conclude in Sec. 6.

2. PRELIMINARIES

Linear Temporal Logic (LTL) formula over a set of *atomic propositions* Σ that can be evaluated as true or false is defined inductively according to the following rules:

- an atomic proposition $\sigma \in \Sigma$ is an LTL formula;
- if φ and ψ are LTL formulas, then also $\neg\varphi$, $\varphi \wedge \psi$, $\bigcirc\varphi$, $\varphi \bigcup \psi$, $\diamond\varphi$, and $\square\varphi$ are LTL formulas,

where \neg (*negation*), \wedge (*conjunction*) are standard Boolean connectives and \bigcirc (*next*), \bigcup (*until*), \diamond (*eventually*), and \square (*always*) are temporal operators. The semantics of LTL is defined over the infinite words over 2^Σ . Informally, $\sigma \in \Sigma$ is satisfied on a word $w = w(1)w(2)\dots$ if $\sigma \in w(1)$. Formula $\bigcirc\varphi$ holds true if φ is satisfied on the word suffix that begins in the next position $w(2)$, whereas $\varphi_1 \bigcup \varphi_2$ states that φ_1 has to be true until φ_2 becomes true. Finally, $\diamond\varphi$ and $\square\varphi$ are true if φ holds on w eventually and always, respectively. For full details, see e.g., [1].

Syntactically co-safe LTL (sc-LTL) is a subclass of LTL built without the *always* operator \square and with the restriction that the *negation* \neg can be applied to atomic propositions only [14]. In contrast to general LTL formulas, the satisfaction of an sc-LTL time can be achieved in a finite time, i.e., each word satisfying an sc-LTL formula φ consists of a *satisfying prefix* that can be followed by an arbitrary suffix.

An *undirected weighted graph* is a tuple $G = (\mathcal{N}, E, W)$, where $\mathcal{N} = \{1, \dots, N\}$ is a set of nodes; $E \subseteq \mathcal{N} \times \mathcal{N}$ is a set of *edges*; and $W : E \rightarrow \mathbb{R}^+$ is the weight function. Each node i has a set of *neighbors* $\mathcal{N}_i = \{j \in \mathcal{N} \mid (i, j) \in E\}$. A path from node i to j is a sequence of nodes starting with i and ending with j such that the consecutive nodes are neighbors. G is *connected* if there is a path between any two nodes and G is *complete* if $E = \mathcal{N} \times \mathcal{N}$. The Laplacian matrix \mathbf{H} of G is an $N \times N$ positive semidefinite matrix: $\mathbf{H}(i, i) = \sum_{j \in \mathcal{N}_i} W(i, j)$, $\forall i \in \mathcal{N}$; $\mathbf{H}(i, j) = W(i, j)$, $\forall (i, j) \in E$, and $\mathbf{H}(i, j) = 0$ otherwise. For a connected graph G , \mathbf{H} has nonnegative eigenvalues [6] and a single zero eigenvalue with the eigenvector $\mathbf{1}_N$, where $\mathbf{1}_N = [1, \dots, 1]^T$.

In this paper, each vector norm over \mathbb{R}^n is the Euclidean norm [10]. We use $|S|$ to denote the cardinality of a set S and $v[i]$ to denote the i -th element of a vector v .

3. PROBLEM FORMULATION

3.1 Agent Dynamics and Network Structure

We consider a team of N autonomous agents with unique identities (IDs) $i \in \mathcal{N} = \{1, \dots, N\}$. They all satisfy the single-integrator dynamics:

$$\dot{x}_i(t) \triangleq u_i(t), \quad i \in \mathcal{N} \quad (1)$$

where $x_i(t)$, $u_i(t) \in \mathbb{R}^2$ are the respective state and the control input of agent i at time $t > 0$, and $x_i(0)$ is the given initial state. The agents are modeled as point masses without volume, i.e., inter-agent collisions are not considered.

Each agent has a sensing radius $r > 0$, which is assumed to be identical for all agents. Namely, each agent can only observe another agent's state if their relative distance is less than r . Thus, given $\{x_i(0), i \in \mathcal{N}\}$, we define the undirected graph $G_0 \triangleq (\mathcal{N}, E_0)$, where $(i, j) \in E_0$ if $\|x_i(0) - x_j(0)\| < r$. We assume that the initial graph G_0 is connected.

3.2 Task Specifications

Within the 2D workspace, each agent $i \in \mathcal{N}$ has a set of M_i regions of interest: $\Pi_i \triangleq \{\varpi_{i1}, \dots, \varpi_{iM_i}\}$. These regions can be of different shapes, such as spheres, triangles, or polygons. For simplicity of presentation, $\varpi_{i\ell} \in \Pi_i$ is here represented by a circular area around a point of interest:

$$\varpi_{i\ell} = \mathcal{B}(c_{i\ell}, r_{i\ell}) = \{y \in \mathbb{R}^2 \mid \|y - c_{i\ell}\| \leq r_{i\ell}\}, \quad (2)$$

where $c_{i\ell} \in \mathbb{R}^2$ is the center; $r_{i\ell} \geq r_{\min}$ is the radius, and $r_{\min} > 0$ is a given minimal radius for all regions. We assume that the regions of interest do not intersect and that the workspace is bounded, which imply the following assumption necessary for the design of the agents' controllers:

ASSUMPTION 1. (I) $\|c_{i\ell_i} - c_{j\ell_j}\| > 2r_{\min}$, $\forall i, j \in \mathcal{N}$, $\forall \varpi_{i\ell_i} \in \Pi_i$ and $\forall \varpi_{j\ell_j} \in \Pi_j$. (II) $\|c_{i\ell}\| < c_{\max}$, $\forall i \in \mathcal{N}$ and $\forall \varpi_{i\ell} \in \Pi_i$, where $c_{\max} > 0$ is a given constant.

Each region of interest is associated with a subset of atomic propositions Σ_i through the labeling function $L_i : \Pi_i \rightarrow 2^{\Sigma_i}$. Without loss of generality, we assume that $\Sigma_i \cap \Sigma_j = \emptyset$, for all $i, j \in \mathcal{N}$ such that $i \neq j$. We view the atomic propositions $L_i(\varpi_{i\ell})$ as *services* that the agent i can provide when being present in region $\varpi_{i\ell} \in \Pi_i$. Hence, upon the visit to $\varpi_{i\ell}$, the agent i chooses among $L_i(\varpi_{i\ell})$ the subset of atomic propositions to be evaluated as true (i.e., the subset of services it provides among the available ones). We denote by $\mathbf{x}_i(T)$ the *trajectory* of agent i during the time interval $[0, T]$, where $T > 0$ and T can be infinity. The trajectory $\mathbf{x}_i(T)$ is associated with a unique finite or infinite sequence $\mathbf{p}_i(T) \triangleq \pi_{i1}\pi_{i2}\dots$ of regions in Π_i that the agent i crosses, and with a finite or infinite sequence of time instants $t'_{i0}t_{i1}t'_{i1}t_{i2}t'_{i2}\dots$ when i enters/leaves the respective regions. Formally, for all $k \geq 1$: $0 = t'_{i0} \leq t_{ik} \leq t'_{ik} < t_{ik+1} < T$, $x_i(t) \in \pi_{ik}$, $\pi_{ik} \in \Pi_i$, $\forall t \in [t_{ik}, t'_{ik}]$, and $x_i(t) \notin \varpi_{i\ell}$, $\forall \varpi_{i\ell} \in \Pi_i$ and $\forall t \in (t'_{ik-1}, t_{ik})$. The *trace* corresponding to $\mathbf{x}_i(T)$ is a sequence of labels of the visited regions $\mathbf{trace}_i(T) \triangleq L_i(\pi_{i1})L_i(\pi_{i2})\dots$, which represents the sequence of atomic propositions that *can* be true (i.e., the services that can be provided) by the agent i following $\mathbf{x}_i(T)$.

The *word* the agent i produces is a sequence of atomic propositions that *actually are* evaluated as true (i.e., the actually provided services). Note that the agent's word and trajectory have to comply: if $\mathbf{trace}_i(T)$ is as above, then $\mathbf{word}_i(T) = w_{\ell_1}w_{\ell_2}\dots$, where $w_{\ell_k} \subseteq L_i(\pi_{\ell_k})$, for all $k \geq 1$.

The specification of the local task for each agent $i \in \mathcal{N}$ is given as a general LTL or an sc-LTL formula φ_i over Σ_i and captures requirements on the services to be provided. In this work, we do not focus on how the service providing is executed by an agent; we only aim at controlling an agent's motion to reach regions where these services are available. Formally, an infinite trajectory $\mathbf{x}_i(T)$ of an agent i satisfies

a given formula φ_i if and only if there exists an infinite word $\text{word}_i(T)$ that complies with $\mathbf{x}_i(T)$ and satisfies φ_i .

3.3 Problem Statement

PROBLEM 1. *Given a team of N agents as in Sec. 3.1, and their task specifications as in Sec. 3.2, design distributed control laws u_i , $\forall i \in \mathcal{N}$, such that for $T \rightarrow \infty$:*

- (1) $\mathbf{x}_i(T)$ satisfies φ_i ; and
- (2) $\|x_i(t) - x_j(t)\| < r$, $\forall (i, j) \in E_0$, $\forall t \in [0, T)$.

4. SOLUTION

The proposed solution consists of three layers: (i) an offline synthesis of a discrete plan, i.e., a sequence of progressive goal regions for each agent; (ii) a distributed continuous control scheme guaranteeing that one of the agents reaches its progressive goal region in finite time while the relative-distance constraints are fulfilled at all time; (iii) a hybrid control layer, which monitors the discrete plan execution and switches between different continuous control modes to achieve the satisfaction of each LTL task.

4.1 Discrete Plan Synthesis

The discrete plan can be generated using standard techniques leveraging ideas from automata-based formal verification. Loosely speaking, an LTL or an sc-LTL formula φ_i is first translated into a Büchi or a finite automaton, respectively. The automaton is viewed as a graph and analyzed using graph search algorithms. As a result, a word that satisfies φ_i is obtained and mapped onto the sequence of regions to be visited. Current temporal logic-based discrete plan synthesis algorithms can accommodate various environmental constraints and advanced plan optimality criteria. We refer the interested reader to related literature, e.g., [1, 2].

It can be shown that without loss of generality, the derived plan of an agent i is in a prefix-suffix form $\tau_i = \tau_{i,\text{pre}}(\tau_{i,\text{suf}})^\omega$, where $\tau_{i,\text{pre}} = (\pi_{i1}, w_{i1}) \dots (\pi_{ik_i}, w_{ik_i})$ is the plan prefix, and $\tau_{i,\text{suf}} = (\pi_{ik_i+1}, w_{ik_i+1}) \dots (\pi_{iK_i}, w_{iK_i})$ is the periodical plan suffix; $\pi_{ik} \in \Pi_i$ and $w_{ik} \subseteq L_i(\pi_{ik})$, $\forall k = 1, \dots, K_i$. Simply speaking, τ_i represents the sequence of *progressive goal regions* $\pi_{i1}\pi_{i2} \dots$ and the word, i.e., the sequence of services to be provided there $w_{i1}w_{i2} \dots$ that satisfies φ_i . If $\{\varphi_i, i \in \mathcal{N}\}$ are all sc-LTL formulas, $\tau_{i,\text{pre}}$ represents the satisfying prefix and the suffix $\tau_{i,\text{suf}}$ can be disregarded.

4.2 Continuous Controller Design

Before stating the proposed control scheme, let us first introduce the notion of connectivity graph, which will allow us to handle the relative-distance constraints. Recall that each agent has a limited sensing radius $r > 0$ as mentioned in Sec. 3.1. Let $\delta \in (0, r)$ be a given constant. Then we define the connectivity graph $G(t)$ as follows:

DEFINITION 1. *Let $G(t) \triangleq (\mathcal{N}, E(t))$ denote the undirected time-varying connectivity graph at time $t \geq 0$, where $E(t) \subseteq \mathcal{N} \times \mathcal{N}$ is the set of edges. (I) $G(0) = G_0$; (II) At time $t > 0$, $(i, j) \in E(t)$ iff one of the following conditions hold: (1) $\|x_i(t) - x_j(t)\| \leq r - \delta$; or (2) $r - \delta < \|x_i(t) - x_j(t)\| \leq r$ and $(i, j) \in E(t^-)$, where $t^- < t$ and $|t - t^-| \rightarrow 0$.*

Note that the condition (II) above guarantees that a new edge will only be added when the distance between two unconnected agents decreases below $r - \delta$. In other words,

there is a hysteresis effect when adding new edges to the connectivity graph. Each agent $i \in \mathcal{N}$ has a time-varying set of neighbors $\mathcal{N}_i(t) = \{j \in \mathcal{N} \mid (i, j) \in E(t)\}$. Note that the graph G_0 defined in Sec. 3.1 is assumed to be connected.

Given that the progressive goal region at time t is $\pi_{ig} = \mathcal{B}(c_{ig}, r_{ig}) \in \Pi_i$, we propose the following two control modes: (1) the *active* mode:

$$\mathbf{C}_{act}: \quad u_i(t) \triangleq -d_i p_i - \sum_{j \in \mathcal{N}_i(t)} h_{ij} x_{ij}, \quad (3)$$

(2) the *passive* mode:

$$\mathbf{C}_{pas}: \quad u_i(t) \triangleq - \sum_{j \in \mathcal{N}_i(t)} h_{ij} x_{ij}, \quad (4)$$

where $x_{ij} \triangleq x_i - x_j$; $p_i \triangleq x_i - c_{ig}$; and the coefficients d_i and h_{ij} are given by

$$d_i \triangleq \frac{\varepsilon^3}{(\|p_i\|^2 + \varepsilon)^2} + \frac{\varepsilon^2}{2(\|p_i\|^2 + \varepsilon)}; h_{ij} \triangleq \frac{r^2}{(r^2 - \|x_{ij}\|^2)^2}, \quad (5)$$

where $\varepsilon > 0$ is a design parameter to be appropriately tuned. We show in detail how to choose ε in the sequel. Note that both controllers in (3) and (4) are nonlinear and rely on only locally-available states: $x_i(t)$ and $\{x_j(t), j \in \mathcal{N}_i(t)\}$.

Assume that $G(T_s)$ is connected at time $T_s > 0$. Moreover, assume that there are $N_a \geq 1$ agents within \mathcal{N} that are in the *active* mode obeying (3) with its goal region as $\pi_{ig} = \mathcal{B}(c_{ig}, r_{ig}) \in \Pi_i$; and the rest $N_p = N - N_a$ agents that are in the *passive* mode obeying (4). For simplicity, denote by the group of active and passive agents $\mathcal{N}_a, \mathcal{N}_p \subseteq \mathcal{N}$ respectively. Note that it is allowed that $N_a = N$ and $N_p = 0$ when all agents are in the active mode.

In the rest of this section, we show that for *any* allowed combination of $N_a > 1$ and $N_p < N$, by following the control laws (3) and (4), *one* active agent reaches its goal region within finite time $T_f \in (T_s, +\infty)$, while $\|x_i(t) - x_j(t)\| < r$, $\forall (i, j) \in E(T_s)$ and $\forall t \in [T_s, T_f]$.

4.2.1 Relative-Distance Maintenance

In this part, we show that the relative-distance constraints are always satisfied by following the control laws (3) and (4) for *any* number of active and passive agents within the system following a potential-field based analysis. We propose the following potential-field function:

$$V(t) \triangleq \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i(t)} \phi_c(x_{ij}) + b_i \sum_{i \in \mathcal{N}} \phi_g(x_i) \quad (6)$$

where $\phi_c(\cdot)$ stands for an attractive potential to agent i 's neighbors and is given by:

$$\phi_c(x_{ij}) \triangleq \frac{1}{2} \frac{\|x_{ij}\|^2}{r^2 - \|x_{ij}\|^2}, \quad \|x_{ij}\| \in [0, r - \delta]; \quad (7)$$

while $\phi_g(\cdot)$ is an attractive force to agent i 's goal defined by:

$$\phi_g(x_i) \triangleq \frac{\varepsilon^2}{2} \frac{\|p_i\|^2}{\|p_i\|^2 + \varepsilon} + \frac{\varepsilon^2}{4} \ln(\|p_i\|^2 + \varepsilon), \quad (8)$$

where function $\ln(\cdot)$ is the natural logarithm; $b_i \in \mathbb{B}$ indicates the agent i 's control mode. Namely, $b_i = 1$, $\forall i \in \mathcal{N}_a$ and $b_i = 0$, $\forall i \in \mathcal{N}_p$. It can be verified that the gradient of

$V(t)$ from (6) with respect to x_i is given by

$$\begin{aligned}\nabla_{x_i} V &= \frac{\partial V}{\partial x_i} = \nabla_{x_i} \phi_g(x_i) + \sum_{j \in \mathcal{N}_i} \nabla_{x_i} \phi_c(x_{ij}) \\ &= b_i d_i p_i + \sum_{j \in \mathcal{N}_i(t)} h_{ij} x_{ij} = -u_i.\end{aligned}\quad (9)$$

THEOREM 1. *$G(t)$ remains connected and no existing edges within $E(T_s)$ will be lost, namely $E(T_s) \subseteq E(t)$, $\forall t \geq T_s$.*

PROOF. Assume that the network $G(t)$ remains *invariant* during the time period $[t_1, t_2] \subseteq [T_s, \infty)$. Thus the neighboring sets $\{\mathcal{N}_i, i \in \mathcal{N}\}$ also remain invariant and $V(t)$ is differentiable for $t \in [t_1, t_2]$. Then the time derivative of $V(t)$ is given by

$$\begin{aligned}\dot{V}(t) &= \sum_{i \in \mathcal{N}} (\nabla_{x_i} V)^T \dot{x}_i = \sum_{i \in \mathcal{N}} (\nabla_{x_i} V)^T u_i \\ &= - \sum_{i \in \mathcal{N}} \|b_i d_i p_i + \sum_{j \in \mathcal{N}_i(t)} h_{ij} x_{ij}\|^2 \leq 0,\end{aligned}\quad (10)$$

meaning that $V(t)$ is non-increasing, $\forall t \geq T_s$. Thus $V(t) \leq V(T_s) < +\infty$ for $t \geq T_s$.

On the other hand, assume a *new* edge (p, q) is added to $G(t)$ at $t = t_2$, where $p, q \in \mathcal{N}$. By Def. 1, $\|x_{pq}(t_2)\| \leq r - \delta$ and $\phi_c(x_{pq}(t_2)) = \frac{(r-\delta)^2}{\delta(2r-\delta)} < +\infty$ since $0 < \varepsilon < r$. Denote by $\hat{E} \subset \mathcal{N} \times \mathcal{N}$ the set of newly-added edges at $t = t_2$. Let $V(t_2^+)$ and $V(t_2^-)$ be the value of $V(t)$ before and after adding the set of new edges to $G(t)$ at $t = t_2$. We get $V(t_2^+) = V(t_2^-) + \sum_{(p,q) \in \hat{E}} \phi_c(x_{pq}(t_2)) \leq V(t_2^-) + |\hat{E}| \frac{(r-\delta)^2}{\varepsilon(2r-\delta)} < +\infty$, where we use the fact that $|\hat{E}|$ is bounded as $\hat{E} \subset \mathcal{N} \times \mathcal{N}$. Thus $V(t) < +\infty$ also holds when new edges are added. Similar analysis can be found in [11].

As a result, $V(t) < +\infty$ for $t \in [T_s, \infty)$. By Def. 1, one existing edge $(i, j) \in E(t)$ will be lost only if $x_{ij}(t) = r$. It implies that $\phi_c(x_{ij}) \rightarrow +\infty$, i.e., $V(t) \rightarrow +\infty$ by (6). By contradiction, we can conclude that new edges might be added but no existing edges will be lost, namely $E(T_s) \subseteq E(t)$, $\forall t \geq T_s$. If $G(T_s)$ is connected, then $G(t)$ remains connected for $\forall t \geq T_s$. It completes the proof. \square

Note that Theorem 1 holds also when $N_a = 0$, i.e., there are no active agents, as (10) still holds when $b_i = 0$, $\forall i \in \mathcal{N}$.

4.2.2 Convergence Analysis

In this part, we aim at analyzing the convergence properties of the closed-loop system. Since we have shown that $V(t)$ is non-increasing for all $t > T_s$ by Theorem 1 above, by LaSalle's invariance principle [12] we only need to find out the largest invariant set that $\dot{V}(t) = 0$, which implies:

$$b_i d_i p_i + \sum_{j \in \mathcal{N}_i(t)} h_{ij} x_{ij} = 0, \quad \forall i \in \mathcal{N}. \quad (11)$$

Then we can construct one $N \times N$ diagonal matrix \mathbf{D} that $\mathbf{D}(i, i) = b_i d_i$, $\forall i \in \mathcal{N}$ and $\mathbf{D}(i, j) = 0$, $i \neq j$ and $i, j \in \mathcal{N}$. and another $N \times N$ matrix \mathbf{H} that $\mathbf{H}(i, i) = \sum_{j \in \mathcal{N}_i} h_{ij}$, $\forall i \in \mathcal{N}$ and $\mathbf{H}(i, j) = -h_{ij}$, $i \neq j$ and $\forall (i, j) \in E(t)$ while $\mathbf{H}(i, j) = 0$, $\forall (i, j) \notin E(t)$. Note that $h_{ij} > 0$ as $\|x_{ij}\| \in [0, r)$ by (9), $\forall (i, j) \in E(t)$. As a result, \mathbf{H} is the Laplacian matrix of the graph $G(t) = (\mathcal{N}, E(t), W)$, where $W(i, j) = h_{ij}$, $\forall (i, j) \in E(t)$. Then (11) can be written in vector form:

$$\mathbf{H} \otimes \mathbf{I}_2 \cdot \mathbf{x} + \mathbf{D} \otimes \mathbf{I}_2 \cdot (\mathbf{x} - \mathbf{c}) = 0, \quad (12)$$

where \otimes is the Kronecker product [10]; \mathbf{x} is the stack vector for x_i , $i \in \mathcal{N}$ and $\mathbf{x}[i] = x_i$; \mathbf{I}_2 is the identity matrix; \mathbf{c} is the stack vector that $\mathbf{c}[i] = c_{ig}$ if $i \in \mathcal{N}_a$ and $\mathbf{c}[i] = \mathbf{0}_2$ if $i \in \mathcal{N}_p$. Let \mathcal{C} be the set of critical points satisfying (12), i.e.,

$$\mathcal{C} \triangleq \{x \in \mathbb{R}^{2N} \mid \mathbf{H} \otimes \mathbf{I}_2 \cdot \mathbf{x} + \mathbf{D} \otimes \mathbf{I}_2 \cdot (\mathbf{x} - \mathbf{c}) = 0\}. \quad (13)$$

Now we show that at the critical points all agent relative distances can be made arbitrarily small by reducing ε and the corresponding connectivity graph is a complete graph.

LEMMA 2. *For all critical points $\mathbf{x}_c \in \mathcal{C}$, (I) $\|x_{ij}\|$ can be made arbitrarily small by reducing ε , $\forall (i, j) \in E(t)$; (II) there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, then the connectivity graph $G(t)$ is complete.*

PROOF. (I) Consider the following equation for $\mathbf{x}_c \in \mathcal{C}$

$$\sum_{(i,j) \in E(t)} h_{ij} \|x_{ij}\|^2 = \mathbf{x}_c^T \cdot (\mathbf{H} \otimes \mathbf{I}_2) \cdot \mathbf{x}_c.$$

Combining the above equation with (12), we get

$$\begin{aligned}\sum_{(i,j) \in E(t)} h_{ij} \|x_{ij}\|^2 &= -\mathbf{x}_c^T \cdot (\mathbf{D} \otimes \mathbf{I}_2) \cdot (\mathbf{x}_c - \mathbf{c}) \\ &= -(\mathbf{x}_c - \mathbf{c})^T \cdot (\mathbf{D} \otimes \mathbf{I}_2) \cdot (\mathbf{x}_c - \mathbf{c}) \\ &\quad - \mathbf{c}^T \cdot (\mathbf{D} \otimes \mathbf{I}_2) \cdot (\mathbf{x}_c - \mathbf{c}) \\ &= - \sum_{i \in \mathcal{N}} b_i d_i (\|p_i\|^2 + c_{ie}^T p_i) \leq \sum_{i \in \mathcal{N}} b_i \|c_{ie}\| d_i \|p_i\|.\end{aligned}\quad (14)$$

Since $d_i \|p_i\| < \varepsilon \sqrt{\varepsilon}$ for $\|p_i\| \geq 0$, we get

$$\sum_{(i,j) \in E(t)} h_{ij} \|x_{ij}\|^2 < N_a c_{\max} \varepsilon \sqrt{\varepsilon} \leq N c_{\max} \varepsilon \sqrt{\varepsilon}, \quad (15)$$

where $\|c_{ie}\| < c_{\max}$ is given in Assump. 1. Thus $\forall (i, j) \in E(t)$, it holds that $h_{ij} \|x_{ij}\|^2 < N c_{\max} \varepsilon \sqrt{\varepsilon} \triangleq \varsigma$. It can be verified that $h_{ij} \|x_{ij}\|^2$ is monotonically increasing as a function of $\|x_{ij}\|$. This implies that $\forall (i, j) \in E(t)$, $\|x_{ij}\|^2 \leq r^2 \varsigma$, or equivalently $\|x_{ij}\|^2 \leq \varepsilon \sqrt{\varepsilon} \xi$, where

$$\xi \triangleq r^2 N c_{\max}. \quad (16)$$

Thus $\|x_{ij}\|$ can be made arbitrarily small by reducing ε .

(II) Moreover, let ε_0 satisfy

$$(N-1) \sqrt{\varepsilon_0 \sqrt{\varepsilon_0} \xi} < r - \delta. \quad (17)$$

If $\varepsilon < \varepsilon_0$, then for any pair $(p, q) \in \mathcal{N} \times \mathcal{N}$, $\|x_{pq}\|$ satisfies

$$\|x_{pq}\| = |x_p - x_1 + x_1 - x_2 + \dots - x_q| \leq (N-1) \sqrt{\varepsilon \sqrt{\varepsilon} \xi} < r - \delta,$$

where we use two facts: there exists a path in $G(t)$ of maximal length N from any node $p \in \mathcal{N}$ to another node q as $G(t)$ remains connected for $t > T_s$ by Lemma 1; and $\|x_{ij}\| \leq \varepsilon \sqrt{\varepsilon} \xi$ from above, $\forall (i, j) \in E(t)$. By Def. 1 this implies $(p, q) \in E(t)$. Thus $G(t)$ is a complete graph. \square

Before stating the convergence property, we need to define the following sets for all $i \in \mathcal{N}_a$:

$$\mathcal{S}_i \triangleq \{\mathbf{x} \in \mathbb{R}^{2N} \mid \|\mathbf{x} - \mathbf{1}_N \otimes c_{ie}\| \leq r_S(\varepsilon)\}, \quad (18)$$

where $r_S(\varepsilon) \triangleq \sqrt{3N\varepsilon} + \sqrt{(N-1)\varepsilon\sqrt{\varepsilon}\xi}$ and ξ is defined in (16). Loosely speaking, \mathcal{S}_i represents the neighbourhood around the goal region center of the active agent $i \in \mathcal{N}_a$. Furthermore, let $\mathcal{S} \triangleq \cup_{i \in \mathcal{N}_a} \mathcal{S}_i$ and $\mathcal{S}^* \triangleq \mathbb{R}^{2N} \setminus \mathcal{S}$.

In the following, we analyze the properties of the critical points within \mathcal{S} and \mathcal{S}^\neg . More specifically: by Lemma 3 there are no local minimal but saddle points within \mathcal{S}^\neg ; by Lemma 4 these saddle points are non-degenerate, (II) by Lemmas 5-7 all critical points within \mathcal{S} are local minima.

To explore these properties, we compute the second partial derivatives of $V(t)$ with respect to x_i , which are given by

$$\begin{aligned} \frac{\partial^2 V}{\partial x_i \partial x_i} &= b_i d_i \otimes \mathbf{I}_2 + b_i d'_i p_i \cdot p_i^T \\ &+ \sum_{j \in \mathcal{N}_i(t)} (h_{ij} \otimes \mathbf{I}_2 + h'_{ij} x_{ij} \cdot x_{ij}^T) \end{aligned} \quad (19)$$

and

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = -h_{ij} \otimes \mathbf{I}_2 - h'_{ij} x_{ij} \cdot x_{ij}^T, \quad \forall j \neq i, \quad (20)$$

where

$$d'_i = \frac{-4\varepsilon^3}{(\|p_i\|^2 + \varepsilon)^3} + \frac{-\varepsilon^2}{(\|p_i\|^2 + \varepsilon)^2}, \text{ and } h'_{ij} = \frac{4r^2}{(r^2 - \|x_{ij}\|^2)^3}. \quad (21)$$

LEMMA 3. *There are no local minima of V within \mathcal{S}^\neg .*

PROOF. We prove this by showing that if a critical point $\mathbf{x}_c \in \mathcal{S}^\neg$ there always exists a direction $\mathbf{z} \in \mathbb{R}^{2N}$ at \mathbf{x}_c such that the quadratic form $\mathbf{z}^T \nabla^2 V \mathbf{z}$ is negative semi-definite.

Given a critical point $\mathbf{x}_c \in \mathcal{C}$ and $\mathbf{x}_c \in \mathcal{S}^\neg$, then by definition $\|\mathbf{x} - \mathbf{1}_N \otimes c_{i\ell}\| > r_s, \forall i \in \mathcal{N}_a$. On the other hand, for any $i \in \mathcal{N}_a$, we can bound $\|\mathbf{x} - \mathbf{1}_N \otimes c_{i\ell}\|$ as follows:

$$\begin{aligned} \|\mathbf{x} - \mathbf{1}_N \otimes c_{i\ell}\| &= \|\mathbf{x} - \mathbf{1}_N \otimes x_i + \mathbf{1}_N \otimes (x_i - c_{i\ell})\| \\ &\leq \sqrt{\sum_{j \in \mathcal{N}} \|x_{ij}\|^2} + \sqrt{N} \|p_i\| \leq \sqrt{(N-1)\varepsilon\sqrt{\varepsilon}\xi} + \sqrt{N} \|p_i\|, \end{aligned}$$

where $\|x_{ij}\|^2 \leq \varepsilon\sqrt{\varepsilon}\xi$ at $\mathbf{x}_c, \forall (i, j) \in E(t)$ by Lemma 2. By comparing it with $r_s(\varepsilon)$, we get $\|p_i\| \geq \sqrt{3\varepsilon}, \forall i \in \mathcal{N}_a$.

Choose $\mathbf{z} \triangleq \mathbf{1}_N \otimes z$, where $z \in \mathbb{R}^2$ and $\|z\| \triangleq 1$. Then $\mathbf{z}^T \nabla^2 V \mathbf{z}$ is evaluated by using (19)-(21):

$$\mathbf{z}^T \nabla^2 V \mathbf{z} = \sum_{i \in \mathcal{N}} b_i d_i z^T z + b_i d'_i z^T p_i p_i^T z \triangleq z^T M z,$$

where $M \triangleq \sum_{i \in \mathcal{N}_a} (d_i \otimes \mathbf{I}_2 + d'_i p_i p_i^T)$ is a 2×2 Hermitian matrix. The trace of M is computed as

$$\begin{aligned} \text{trace}(M) &= \sum_{i \in \mathcal{N}_a} 2d_i + d'_i \|p_i\|^2 \\ &= \varepsilon^3 \sum_{i \in \mathcal{N}_a} \frac{3\varepsilon - \|p_i\|^2}{(\|p_i\|^2 + \varepsilon)^3} < 0, \end{aligned} \quad (22)$$

as we have shown that $\|p_i\| \geq \sqrt{3\varepsilon}, \forall i \in \mathcal{N}_a$ if $\mathbf{x}_c \in \mathcal{S}^\neg$. On the other hand, denote by $p_i = [p_{i,x}, p_{i,y}]$ the coordinates of p_i . The determinant of M is given by

$$\begin{aligned} \det(M) &= -(\sum_{i \in \mathcal{N}_a} d'_i p_{i,x} p_{i,y})^2 \\ &+ (\sum_{i \in \mathcal{N}_a} d_i + d'_i p_{i,x}^2)(\sum_{i \in \mathcal{N}_a} d_i + d'_i p_{i,y}^2) \\ &\geq \frac{1}{2} \sum_{i, j \in \mathcal{N}_a} [(d_i + d'_i \|p_i\|^2)(d_j + d'_j \|p_j\|^2)] > 0, \end{aligned} \quad (23)$$

since $d'_i \|p_i\|^2 < -d_i$ for $\|p_i\| > \sqrt{3\varepsilon}, \forall i \in \mathcal{N}_a$; and $(p_{i,x} p_{i,y} - p_{j,x} p_{j,y})^2 \leq \|p_i\|^2 \|p_j\|^2$ by Cauchy-Schwarz inequality [10].

Denote by λ_1 and λ_2 the eigenvalues of M , where $\lambda_1, \lambda_2 \in \mathbb{R}$ as M is Hermitian. Since $\text{trace}(M) < 0$ and $\det(M) > 0$, then M is negative definite and both eigenvalues are negative [10], i.e., $\lambda_1, \lambda_2 < 0$. Thus for any vector $v = \mathbf{1}_N \otimes z$ where $z \in \mathbb{R}^2, v^T \nabla^2 V v < 0$. To conclude, for any critical point $\mathbf{x}_c \in \mathcal{C}$, if $\mathbf{x}_c \in \mathcal{S}^\neg$ then \mathbf{x}_c is not a local minimum. \square

LEMMA 4. *There exists $\varepsilon_1 > 0$ such that if $\varepsilon < \varepsilon_1$, all critical points of V in \mathcal{S}^\neg are non-degenerate saddle points.*

PROOF. To show that V is Morse we use Lemma 3.8 from [19], which states that the non-singularity of a linear operator follows from the fact that its associated quadratic form is sign definite on complementary subspaces.

Let $\mathcal{Q} = \{v \in \mathbb{R}^{2N} \mid v = \mathbf{1}_N \otimes z, z \in \mathbb{R}^2\}$. In Lemma 3, we have shown that for any vector $v \in \mathcal{Q}, v^T \nabla^2 V v < 0$. Let $\mathcal{P} = \{v \in \mathbb{R}^{2N} \mid v = \mathbf{e}_N \otimes z, \mathbf{e}_N \perp \mathbf{1}_N, \mathbf{e}_N \in \mathbb{R}^N, z \in \mathbb{R}^2\}$. Firstly, it can be easily verified that \mathcal{P} is the orthogonal complement of \mathcal{Q} . In the following, we show that $\nabla^2 V$ is positive definite in \mathcal{P} . Let $\mathbf{z} \in \mathcal{P}$, i.e., $\mathbf{z} \triangleq \mathbf{e}_N \otimes z \triangleq [z_1^T z_2^T \dots z_n^T]^T$, where $z \in \mathbb{R}^2, \mathbf{e}_N \in \mathbb{R}^N, \mathbf{e}_N^T \perp \mathbf{1}_N, z_i \in \mathbb{R}^2, \forall i \in \mathcal{N}$. The quadratic form $\mathbf{z}^T \nabla^2 V \mathbf{z}$ at \mathbf{x}_c is computed using (19)-(21):

$$\begin{aligned} \mathbf{z}^T \nabla^2 V \mathbf{z} &= \sum_{i \in \mathcal{N}_a} (d_i \|z_i\|^2 + d'_i |p_i^T z_i|^2) \\ &+ \sum_{(i, j) \in E(t)} (h_{ij} \|z_i - z_j\|^2 + 2h'_{ij} |(x_i - x_j)^T (z_i - z_j)|^2) \\ &\geq \sum_{i \in \mathcal{N}_a} (d_i \|z_i\|^2 + d'_i |p_i^T z_i|^2) + \sum_{(i, j) \in E(t)} h_{ij} \|z_i - z_j\|^2 \\ &\geq \sum_{i \in \mathcal{N}_a} (d_i + d'_i \|p_i\|^2) \|z_i\|^2 + \mathbf{z}^T (\mathbf{H} \otimes \mathbf{I}_2) \mathbf{z}, \end{aligned}$$

where we use the fact that $h'_{ij} > 0, d'_i < 0$ and $|p_i^T z_i| \leq \|p_i\| \|z_i\|$. It can be verified that $d_i + d'_i \|p_i\|^2 > -0.1\varepsilon$ for $\|p_i\| \geq \sqrt{3\varepsilon}, \forall i \in \mathcal{N}_a$. Moreover,

$$\begin{aligned} \mathbf{z}^T (\mathbf{H} \otimes \mathbf{I}_2) \mathbf{z} &= (\mathbf{e}_N \otimes z)^T \cdot (\mathbf{H} \otimes \mathbf{I}_2) \cdot (\mathbf{e}_N \otimes z) \\ &= (\mathbf{e}_N^T \cdot \mathbf{H} \cdot \mathbf{e}_N) \|z\|^2 \geq \lambda_2(\mathbf{H}) \|z\|^2, \end{aligned} \quad (24)$$

where we apply the Courant-Fischer Theorem [10]:

$$\min_{\mathbf{e}_N \perp \mathbf{1}_N} \{\mathbf{e}_N^T \cdot \mathbf{H} \cdot \mathbf{e}_N\} = \lambda_2(\mathbf{H}) > 0,$$

since \mathbf{H} is the Laplacian matrix defined in (12), which is positive semidefinite with $\lambda_1(\mathbf{H}) = 0$, of which the corresponding eigenvector is $\mathbf{1}_N$; and the second smallest eigenvalue $\lambda_2(\mathbf{H}) > 0$. In addition, since $h_{ij} > 1/r^2$ and $G(t)$ is a complete graph at \mathbf{x}_c by Lemma 2, it holds that $\lambda_2(\mathbf{H}) > N/r^2$ by [6]. This implies that

$$\begin{aligned} \mathbf{z}^T \nabla^2 V \mathbf{z} &\geq \sum_{i \in \mathcal{N}_a} \left(\frac{N}{r^2} + d_i + d'_i \|p_i\|^2 \right) \|z_i\|^2 \\ &\geq \sum_{i \in \mathcal{N}_a} \left(\frac{N}{r^2} - 0.1\varepsilon \right) \|z_i\|^2. \end{aligned} \quad (25)$$

Thus if $\varepsilon < N/(0.1r^2)$, it holds that $\mathbf{z}^T \nabla^2 V \mathbf{z} > 0, \forall \mathbf{z} = \mathbf{e}_N \otimes z$ where $\mathbf{e}_N \perp \mathbf{1}_N, z \in \mathbb{R}^2$.

To conclude, $\nabla^2 V|_{\mathcal{Q}}$ is negative definite by Lemma 3 and $\nabla^2 V|_{\mathcal{P}}$ is positive definite by the analysis above. By applying Lemma 3.8 from [19], we can conclude that $\nabla^2 V$ is non-singular at the saddle points $\mathbf{x}_c \in \mathcal{S}^\neg$, if

$$\varepsilon < \min\{\varepsilon_0, \frac{N}{0.1r^2}\} \triangleq \varepsilon_1. \quad (26)$$

In other words, all critical points within \mathcal{S}^\neg are non-degenerate saddle points if $\varepsilon < \varepsilon_1$. \square

Now we focus on proving that all critical points within \mathcal{S} are stable local minima. First of all, we need the following two lemmas to show that when a critical point belongs to \mathcal{S}_i corresponding to one active agent $i \in \mathcal{N}_a$, then all the other agents are within its goal region π_{i^*g} and away from their own goal region center by at least distance r_{\min} .

LEMMA 5. *There exists $\varepsilon_2 > 0$, such that if $\varepsilon < \varepsilon_2$, the following hold: (I) $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, $\forall i \neq j$ and $i, j \in \mathcal{N}_a$; (II) If $\mathbf{x}_c \in \mathcal{S}_{i^*}$ for some $i^* \in \mathcal{N}_a$, then $x_j \in \pi_{i^*g}$, $\forall j \in \mathcal{N}$ and $\|x_j - c_{jg}\| > r_{\min}$, $j \neq i^*$, $\forall j \in \mathcal{N}_a$.*

PROOF. Let ε_2 be given as the solution of

$$r_S(\varepsilon_2) = \sqrt{3N\varepsilon_2} + \sqrt{(N-1)\varepsilon_2\sqrt{\varepsilon_2}\xi} \triangleq r_{\min}, \quad (27)$$

where r_{\min} is given in Assump. 1. Note that (27) has a unique solution as the left-hand side is a function of ε_2 that monotonically increases and has the range $[0, \infty)$.

Assume that $\mathbf{x}_c \in \mathcal{S}_{i^*}$ for $i^* \in \mathcal{N}_a$, i.e., $\|\mathbf{x}_c - \mathbf{1}_N \otimes c_{i^*g}\| \leq r_S(\varepsilon_2)$. Then $\forall j \neq i^*$, $j \in \mathcal{N}_a$, it holds that (I) $\|\mathbf{x}_c - \mathbf{1}_N \otimes c_{jg}\| = \|\mathbf{x}_c - \mathbf{1}_N \otimes c_{ig} + \mathbf{1}_N \otimes c_{ig} - \mathbf{1}_N \otimes c_{jg}\| \geq \sqrt{N}\|c_{ig} - c_{jg}\| - \|\mathbf{x}_c - \mathbf{1}_N \otimes c_{ig}\| \geq 2\sqrt{N}r_{\min} - r_S(\varepsilon)$, due to that $\|c_{ig} - c_{jg}\| > 2r_{\min}$ by Assump. 1. Since $\varepsilon < \varepsilon_2$, then $r_S(\varepsilon) < r_S(\varepsilon_2) = r_{\min}$. Thus $\|\mathbf{x}_c - \mathbf{1}_N \otimes c_{jg}\| > 2\sqrt{N}r_{\min} - r_{\min} > r_{\min} = r_S(\varepsilon_2)$, implying that $\mathbf{x}_c \notin \mathcal{S}_j$. (II) $\|x_j - c_{i^*g}\| < \|\mathbf{x}_c - \mathbf{1}_N \otimes c_{i^*g}\| < r_{\min} < r_{i^*g}$, meaning that $x_j \in \pi_{i^*g}$, $\forall j \in \mathcal{N}$. Moreover, $\|x_j - c_{jg}\| = \|x_j - c_{i^*g} + c_{i^*g} - c_{jg}\| \geq \|c_{i^*g} - c_{jg}\| - \|x_j - c_{i^*g}\| \geq 2r_{\min} - r_{\min} > r_{\min}$. \square

LEMMA 6. *There exists $\varepsilon_6 > 0$ such that if $\varepsilon < \varepsilon_6$, then for a critical point $\mathbf{x}_c \in \mathcal{S}_{i^*}$, $i^* \in \mathcal{N}_a$, then it holds that $\|p_{i^*}\| < \sqrt{0.4\varepsilon}$.*

PROOF. By summing (11) for all $i \in \mathcal{N}$, we get

$$d_{i^*} p_{i^*} = - \sum_{j \neq i^*, j \in \mathcal{N}_a} d_j p_j. \quad (28)$$

Consider the scalar function $f(\|p_j\|) = d_j(\|p_j\|)\|p_j\|$ for $\|p_j\| \geq 0$. It is monotonically increasing for $\|p_j\| \in [0, 3.2\sqrt{\varepsilon})$ and decreasing for $\|p_j\| \in [3.2\sqrt{\varepsilon}, \infty)$.

If $\mathbf{x}_c \in \mathcal{S}_{i^*}$ for $i^* \in \mathcal{N}_a$, then $\|\mathbf{x}_c - \mathbf{1}_N \otimes c_{i^*g}\| \leq r_S(\varepsilon_2)$. Moreover, $\|\mathbf{x} - \mathbf{1}_N \otimes c_{i^*g}\| \geq \|\mathbf{1}_N \otimes x_{i^*} - \mathbf{1}_N \otimes c_{i^*g}\| - \|\mathbf{x} - \mathbf{1}_N \otimes x_{i^*}\| \geq \sqrt{N}\|p_{i^*}\| - \sqrt{(N-1)\varepsilon\sqrt{\varepsilon}\xi}$. This implies $\|p_{i^*}\| \leq \sqrt{3\varepsilon + 2\sqrt{\varepsilon\sqrt{\varepsilon}\xi}}$. Moreover by Lemma 5, $\|p_j\| > r_{\min}$, $\forall j \neq i^*$, $j \in \mathcal{N}_a$. Thus if $r_{\min} > 3.2\sqrt{\varepsilon}$, namely

$$\varepsilon < 0.07 r_{\min}^2 \triangleq \varepsilon_3, \quad (29)$$

it holds that $d_j \|p_j\| < 0.5\varepsilon^2/r_{\min}$, $\forall j \neq i^*$, $j \in \mathcal{N}_a$. Thus $d_{i^*} \|p_{i^*}\| < 0.5(N_a - 1)\varepsilon^2/r_{\min}$ by (28). If the following two conditions hold: (i) $\sqrt{3\varepsilon + 2\sqrt{\varepsilon\sqrt{\varepsilon}\xi}} < 3.2\sqrt{\varepsilon}$; (ii) $0.5(N_a - 1)\varepsilon^2/r_{\min} < d_j(\sqrt{0.4\varepsilon})\sqrt{0.4\varepsilon}$, then $\|p_{i^*}\| < \sqrt{0.4\varepsilon}$ since it is shown earlier that function $d_j(\|p_j\|)\|p_j\|$ is monotonically increasing for $\|p_j\| \in [0, 3.2\sqrt{\varepsilon})$. Condition (i) above implies that $\varepsilon < 4.1/\xi^2 \triangleq \varepsilon_4$ and condition (ii) holds for all $N_a \leq N$ if $\varepsilon < 0.8 r_{\min}^2/(N-1)^2 \triangleq \varepsilon_5$. To conclude, if $\varepsilon < \varepsilon_6$, where

$$\varepsilon_6 \triangleq \min\{\varepsilon_3, \varepsilon_4, \varepsilon_5\}, \quad (30)$$

then $\mathbf{x}_c \in \mathcal{S}_{i^*}$ implies $\|p_{i^*}\| < \sqrt{0.4\varepsilon}$. \square

With the above two lemmas, we can now show that all critical points within \mathcal{S} are stable local minima.

LEMMA 7. *There exists $\varepsilon_{\min} > 0$ such that if $\varepsilon < \varepsilon_{\min}$, all critical points of V within \mathcal{S} are local minima.*

PROOF. A critical point $\mathbf{x}_c \in \mathcal{S}$ can only belong to one set \mathcal{S}_i for $i \in \mathcal{N}_a$ by Lemma 5. Without loss of generality, let $\mathbf{x}_c \in \mathcal{S}_{i^*}$, where $i^* \in \mathcal{N}_a$. In the following we show that \mathbf{x}_c is a local minimum.

Let $\mathbf{z} \in \mathbb{R}^{2N}$ and $\|\mathbf{z}\| = 1$. Set $\mathbf{z} = [z_1^T z_2^T \dots z_n^T]^T$, where $z_i \in \mathbb{R}^2$, $\forall i \in \mathcal{N}$. Then $\mathbf{z}^T \nabla^2 V \mathbf{z}$ at \mathbf{x}_c is computed as:

$$\begin{aligned} \mathbf{z}^T \nabla^2 V \mathbf{z} &= \sum_{i \in \mathcal{N}_a} (d_i \|z_i\|^2 + d'_i |p_i^T z_i|^2) + \\ &\sum_{(i,j) \in E(t)} (h_{ij} \|z_{ij}\|^2 + 2h'_{ij} |x_{ij}^T z_{ij}|^2). \end{aligned} \quad (31)$$

where $z_{ij} \triangleq z_i - z_j$. Since $|p_i^T z_i| \leq \|p_i\| \|z_i\|$, $d_i > 0$ and $d'_i < 0$, it holds that $d_i \|z_i\|^2 + d'_i |p_i^T z_i|^2 \geq (d_i + d'_i \|p_i\|^2) \|z_i\|^2$, $\forall i \in \mathcal{N}_a$. It can be verified that for $j \neq i^*$ and $\forall j \in \mathcal{N}_a$, $d_j + d'_j \|p_j\|^2 > \varepsilon^2 \hat{g}$ where $\hat{g} \triangleq -2/r_{\min}^2$, since $\|p_j\| > r_{\min}$ by Lemma 5; and $d_{i^*} + d'_{i^*} \|p_{i^*}\|^2 > 0.08\varepsilon$ since $\|p_{i^*}\| > \sqrt{0.4\varepsilon}$ by Lemma 6. Regarding the second term of (31), since Lemma 2 shows that $G(t)$ is a complete graph at \mathbf{x}_c with $h_{ij} > 1/r^2$ and $h'_{ij} > 0$, we get

$$\begin{aligned} &\sum_{(i,j) \in E} (h_{ij} \|z_{ij}\|^2 + 2h'_{ij} |x_{ij}^T z_{ij}|^2) \\ &\geq \sum_{j \in \mathcal{N}} h_{i^*j} \|z_{i^*j}\|^2 \geq \frac{1}{r^2} \sum_{j \in \mathcal{N}} \|z_{i^*j}\|^2. \end{aligned} \quad (32)$$

Thus (31) can be bounded by

$$\begin{aligned} \mathbf{z}^T \nabla^2 V \mathbf{z} &\geq \sum_{i \in \mathcal{N}_a} (d_i + d'_i \|p_i\|^2) \|z_i\|^2 + \sum_{j \in \mathcal{N}} h_{i^*j} \|z_{i^*j}\|^2 \\ &\geq 0.08\varepsilon \|z_{i^*}\|^2 - \varepsilon^2 \sum_{j \neq i^*, j \in \mathcal{N}_a} |\hat{g}| \|z_j\|^2 + \frac{1}{r^2} \sum_{j \in \mathcal{N}} \|z_{i^*j}\|^2 \\ &\geq \sum_{j \in \mathcal{N}_a} \left(\frac{1}{r^2} + \frac{0.08\varepsilon}{N} \right) \|z_{i^*}\|^2 + \left(\frac{1}{r^2} - \varepsilon^2 |\hat{g}| \right) \|z_j\|^2 - \frac{2}{r^2} z_{i^*}^T z_j, \end{aligned}$$

as $1 \leq N_a \leq N$. If the following condition holds:

$$\left(\frac{1}{r^2} + \frac{0.08\varepsilon}{N} \right) \left(\frac{1}{r^2} - \varepsilon^2 |\hat{g}| \right) > \left(\frac{1}{r^2} \right)^2, \quad (33)$$

it implies $\mathbf{z}^T \nabla^2 V \mathbf{z} > (|z_{i^*}^T z_j| - z_{i^*}^T z_j)/r^2 \geq 0$, $\forall \mathbf{z} \in \mathbb{R}^{2N}$. Namely, $\nabla^2 V$ is positive definite at critical points $\mathbf{x}_c \in \mathcal{S}$. Condition (33) is equivalent to

$$\varepsilon^2 + \frac{N}{0.08 r^2} \varepsilon - \frac{1}{r^2 |\hat{g}|} < 0.$$

Since $\varepsilon > 0$, this implies that

$$0 < \varepsilon < \frac{\sqrt{(\frac{N}{0.08 r^2})^2 + \frac{4}{r^2 |\hat{g}|}} - \frac{N}{0.08 r^2}}{2} \triangleq \varepsilon_7, \quad (34)$$

To conclude, if

$$\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_6, \varepsilon_7\} \triangleq \varepsilon_{\min}, \quad (35)$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_6$ and ε_7 are defined in (26), (27), (30) and (34), then all local minima within \mathcal{S} are stable. \square

By summarizing Lemmas 3-7, we can derive the following convergence result:

THEOREM 8. Assume that $G(T_s)$ is connected and $\varepsilon < \varepsilon_{\min}$ by (35). Then starting from anywhere in the workspace except a set of measure zero, there exists a finite time $T_f \in [T_s, \infty)$ and one agent $i^* \in \mathcal{N}_a$, such that $x_j(T_f) \in \pi_{i^*g}$, $\forall j \in \mathcal{N}$, while at the same time $\|x_i(t) - x_j(t)\| < r$, $\forall (i, j) \in E(T_s)$ and $\forall t \in [T_s, T_f]$.

PROOF. First of all, the second part follows directly from Theorem 1 which guarantees that all edges within $E(T_s)$ will be reserved for all $t > T_s$. Secondly, we have shown that $V(t)$ by (6) is non-increasing for all $t > T_s$ by Theorem 1. By LaSalle's invariance principle [12] we only need to find out the largest invariant set within $\dot{V}(t) = 0$. Theorems 3 and 7 ensure that the potential function $V(t)$ has only local minima inside \mathcal{S} and saddle points outside \mathcal{S} . These saddle points have attractors of measure zero by Lemma 4. Thus starting from anywhere in the workspace except a set of measure zero, the system converges to the set of local minima. Part (I) of Lemma 5 shows that a local minimum can not belong to two different \mathcal{S}_i simultaneously. Thus the system converges to the set of local minima within \mathcal{S}_{i^*} for one active agent $i^* \in \mathcal{N}_a$. By part (II) of Lemma 5, all agents would enter π_{i^*g} , i.e., $x_j \in \pi_{i^*g}$, $\forall j \in \mathcal{N}$. Consequently, there exists $T_f < \infty$ that $x_j(T_f) \in \pi_{i^*g}$, $\forall j \in \mathcal{N}$, for exactly one active agent $i^* \in \mathcal{N}_a$. \square

REMARK 1. Note that Theorem 8 holds for any number of active agents with $1 \leq N_a \leq N$. In other words, independent of the number of active agents within the team, one active agent will reach its goal region first within finite time, while fulfilling the relative-distance constraints.

4.3 Hybrid Control Structure

In Sec. 4.1, we have generated a sequence of progressive goal regions for each agent and in Sec. 4.2, we have shown that under the proposed control laws all agents converge to one active agent's progressive goal region. In this part, we propose a local procedure for each agent to decide on its own activity/passivity. Thus we integrate the discrete plans and the continuous control laws into a hybrid control scheme to guarantee that every agent's local task is fulfilled.

4.3.1 Switching Protocol for sc-LTL

Let us first focus on the case when each task φ_i , $i \in \mathcal{N}$ is an sc-LTL formula. As introduced in Sec. 4.1, the discrete plan τ_i for agent i can be represented by a finite satisfying prefix of progressive goal regions in Π_i of length $k_i > 0$:

$$\tau_{i,\text{pre}} = (\pi_{i1}, w_{i1}) \dots (\pi_{ik_i}, w_{ik_i})$$

We propose the following *activity switching protocol* for each agent $i \in \mathcal{N}$:

- (I) At time $t = 0$, agent i sets $\varkappa_i := 1$ and itself as active and sets $\pi_{ig} := \pi_{i\varkappa_i}$, namely the first goal region in τ_i . The *active* controller (3) is applied, where the progressive goal region is π_{ig} .
- (II) Whenever agent i reaches its current progressive goal region $\pi_{ig} = \pi_{i\varkappa_i}$ and $\varkappa_i < k_i$, it provides the prescribed set of services $w_{i\varkappa_i}$ and it sets $\varkappa_i := \varkappa_i + 1$ and $\pi_{ig} := \pi_{i\varkappa_i}$. The controller (3) for agent i is updated while the other agents' controllers remain unchanged.
- (III) Whenever agent i reaches its last progressive goal region $\pi_{ig} = \pi_{ik_i}$, it provides the set of services w_{ik_i} by which it finishes the execution of its discrete plan. Afterwards it remains *passive* and controller (4) applies.

THEOREM 9. By following the protocol above, it is guaranteed that $\forall i \in \mathcal{N}$, φ_i is satisfied by $\mathbf{x}_i(T)$, and $\|x_i(t) - x_j(t)\| < r$, $\forall (i, j) \in E(0)$ and $\forall t \geq 0$, where $T \rightarrow \infty$.

PROOF. At $t = 0$, all agents are active and following the controller (3). By Theorem 8, all agents converge to one agent's goal region at a finite time $t_1 > 0$. Denote by $i \in \mathcal{N}$ this agent. Then either by step (II) of the protocol, the agent i updates its active control law by setting $\pi_{ig} = \pi_{i2}$, or by step (III) the agent i has completed its plan $\tau_{i,\text{pre}}$ and becomes passive. Since all agents' plans are finite and Theorem 8 holds for any number of active agents, we obtain that there exists a finite time instant T_{f_j} , at which one of the agents $j \in \mathcal{N}_a$ finishes executing its plan $\tau_{j,\text{pre}}$, i.e., such that φ_j becomes satisfied. Then by step (III), this agent is passive and following the controller (4) for all times $t \in [T_{f_j}, \infty)$. Inductively, we conclude that there exists a time instant T_f , by which all agents complete their plans and all formulas are satisfied. All agents are passive for all $t \in (T_f, \infty)$ and by controller (4) they all converge to one point. The second part of the theorem follows directly from Theorem 8. \square

Note that this protocol is fully decentralized as the decisions on an agents' activity/passivity are local and do not depend on any relative-state measurements.

4.3.2 Switching Protocol for full LTL

As introduced in Sec. 4.1, if the task specification φ_i is given as a general LTL formula, then the plan τ_i is represented by an infinite sequence of progressive goal regions in a prefix-suffix form

$$\begin{aligned} \tau_i &= \tau_{i,\text{pre}}(\tau_{i,\text{suf}})^\omega = (\pi_{i1}, w_{i1})(\pi_{i2}, w_{i2}) \dots, \text{ where} \\ \tau_{i,\text{pre}} &= (\pi_i, w_{i1}) \dots (\pi_{ik_i}, w_{ik_i}), k_i > 0 \text{ and} \\ \tau_{i,\text{suf}} &= (\pi_{ik_i+1}, w_{ik_i+1}) \dots (\pi_{iK_i}, w_{iK_i}), K_i > 0. \end{aligned}$$

The main challenge in this case is to ensure that each agent visits its progressive goal region infinitely often. The activity switching protocol from Sec. 4.3.1 could not be applied here since all agents would remain active at all times. As a result, the team may repetitively converge to π_{ig} for some $i \in \mathcal{N}$ while never visiting the other agents' progressive goal regions (see Sec. 5 for an example). Hence, we aim to design a "fair" activity switching protocol that enforces a progress towards each agent's task. Thereto, we first introduce a communication-free reaching-event detector that enables an agent to monitor its neighbors' plan executions.

Reaching-Event Detector. Agent $i \in \mathcal{N}$ can detect when it reaches its own progressive goal region π_{ig} by checking if $x_i(t) \in \pi_{ig}$. For our switching protocol presented below, it is also essential that it can detect when another agent $j \in \mathcal{N}$ reaches π_{jg} . Note that by Lemma 2, the connectivity graph is complete since the first time any agent $i \in \mathcal{N}$ reaches its progressive goal region π_{ig} , hence it is sufficient to detect when a neighboring agent $j \in \mathcal{N}_i(t)$ reaches π_{jg} .

Given that the agents satisfy the dynamics by (1) and that each agent $i \in \mathcal{N}$ can measure $x_i(t) - x_j(t)$ for all its neighbors $j \in \mathcal{N}_i(t)$ in real time, we assume that the agent i can measure or estimate $u_j(t)$, for all $j \in \mathcal{N}_i(t)$ [5]. Let $\Omega_i(j, t) \in \mathbb{B}$ be a Boolean variable indicating that agent i detects its neighboring agent $j \in \mathcal{N}_i(t)$ reaching the goal region π_{jg} at time $t > 0$. We propose the following reaching-event detector inspired by [17]. Simply speaking, the detector checks if within a short time period $[t - \Delta_t, t]$, there

exists $j \in \mathcal{N}_i(t)$, such that $u_j(t)$ has changed from a relatively small value (below a given Δ_u) by a difference larger than certain Δ_d . If so, it means that the agent j has reached its progressive goal region π_{jg} .

The choice of this reaching-event detector is motivated by the following facts: By (11), all control inputs $u_i(t)$ are close to zero when the system is close to a local minimal, $\forall i \in \mathcal{N}$. Afterwards, our switching protocol introduced below guarantees that *only* agent j switches its control law either to (3) in order to navigate to the next progressive goal region or to (4) in order to become passive. This change is lower-bounded by constant Δ_d derived using control law (3) and Lemmas 5, 6 as $\Delta_d \triangleq |f(r_{\min}) - f(\sqrt{0.4\epsilon})|$, where $f(\|p_j\|) = d_j(\|p_j\|)\|p_j\|$ is a scalar function and $d_j(\|p_j\|)$ is defined by (5). In contrast, for the other agents $i \neq j$, $i \in \mathcal{N}$, the control input $u_i(t)$ remains unchanged and close to zero. Hence, agent j is identified as the only one who has reached its progressive goal region. Formally,

DEFINITION 2. $\Omega_i(j, t) \triangleq \text{True}$ if and only if there exists $t' \in [t - \Delta_t, t]$, where $|u_j(t')| < \Delta_u$ and $|u_j(t) - u_j(t')| > \Delta_d$.

Activity Switching Protocol. Loosely speaking, in the proposed protocol, an agent $i \in \mathcal{N}$ becomes passive if it has made a certain progress towards the satisfaction of its specification, hence giving the other agents an opportunity to advance in execution of their plans. However, once each agent has achieved certain progress, the agent i becomes active again to proceed with its infinite plan. We define a *round* as the time period during which each agent has reached at least one of its goal regions according to their plans.

DEFINITION 3. For all $m \geq 1$, the m -th round is defined as the time interval $[T_{\odot_{m-1}}, T_{\odot_m})$, where $T_{\odot_0} = 0$, $T_{\odot_{m-1}} < T_{\odot_m}$ and for all $m \geq 1$, T_{\odot_m} is the smallest time satisfying the following conditions $\forall i \in \mathcal{N}$: $\text{word}_i(T_{\odot_m}) = w_{i1} \dots w_{i\ell}$ for some $\ell \geq 1$ and $\text{word}_i(T_{\odot_m}) \neq \text{word}_i(T_{\odot_{m-1}})$.

The notion of a round is crucial to the design of the activity switching protocol. To recognize a round completion, we introduce the following variables: $\chi_i \geq 0$ indicates the starting time of the current round, and $\Upsilon_i \in \mathbb{Z}^N$ is a vector to record how many progressive goal regions each agent has reached within one round since χ_i . Although these variables are locally maintained by each agent. By Lemma 2, the connectivity graph is complete since the first time one active agent reaches its goal region π_{ig} and under the assumption of unbiased measurements, it holds that at the same time instant $\chi_i = \chi_j$, and $\Upsilon_i = \Upsilon_j$, $\forall i, j \in \mathcal{N}$. We propose the following *activity switching protocol* for each agent $i \in \mathcal{N}$:

- (I) At time $t = 0$, $\Upsilon_i := \mathbf{0}_N$, $\chi_i := 0$, $\varkappa_i := 1$. The agent i is active and follows control law (3), where $\pi_{ig} := \pi_{i\varkappa_i}$.
- (II) Whenever the agent i reaches its current progressive goal region $\pi_{ig} = \pi_{i\varkappa_i}$ and waits until $|u_i(t)| < \Delta_u$, it provides the prescribed set of services $w_{i\varkappa_i}$ and updates the current progressive goal region accordingly: If $\varkappa_i < K_i$ then $\varkappa_i := \varkappa_i + 1$, and if $\varkappa_i = K_i$ then $\varkappa_i := k_i + 1$. Furthermore, $\pi_{ig} := \pi_{i\varkappa_i}$, and finally $\Upsilon_i[\bar{i}] := \Upsilon_i[\bar{i}] + 1$.

Generally speaking, the agent i decides to stay active or to become passive based on the probability function:

$$\Pr(b_i = 1) = \begin{cases} f_{\text{prob}}(\cdot) & \text{if } f_{\text{cond}}(\cdot) = \text{True}, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{\text{prob}}(\cdot) \in [0, 1]$ and $f_{\text{cond}}(\cdot) \in \{\text{True}, \text{False}\}$ are functions of time t and the local variables Υ_i and χ_i , subject to the following: given that the current round is the m -th one, there exists a time $T \in (T_{\odot_{m-1}}, T_{\odot_m})$, such that $f_{\text{cond}}(\cdot) = \text{False}$ for all $t \in [T, T_{\odot_m})$.

Whenever $b_i = 1$, the agent i keeps following the control law (3) with the updated π_{ig} . Otherwise, it becomes passive and the control law (4) is applied.

- (III) Whenever agent i detects that $\Omega_i(j, t) = \text{True}$, for some $j \neq i \in \mathcal{N}$, it sets $\Upsilon_i[j] = \Upsilon_i[j] + 1$.
- (IV) Whenever for all $j \in \mathcal{N}$ it holds that $\Upsilon_i[j] > 0$ the agent i sets $\Upsilon_i := \mathbf{0}_N$, $\chi_i := t$ and follows the active control law (3).

A straightforward example of the functions choice in (II) is $f_{\text{cond}} = \text{False}$, for all $t \geq 0$. Then the agent i always becomes passive once it visits π_{ig} . Note that it becomes active after the current round is completed by step (IV). However, a different selection of the functions may allow for trading the fairness of activity switching for the increased efficiency of plan executions. The switching to passive control mode may be temporarily postponed and as a result, the visits to progressive goal regions may become more frequent. An example of such a case is given in Sec. 5.

LEMMA 10. The round $[T_{\odot_{m-1}}, T_{\odot_m})$ is finite, $\forall m \geq 1$.

PROOF. Let $t = T_{\odot_{m-1}} = 0$, and thus $\Upsilon_i[j] = 0$, for all $i, j \in \mathcal{N}$ by step (I). By Theorem 8, one of the agents reaches its progressive goal region in finite time at $t_1 \geq T_{\odot_{j-1}}$. Since there are only finite number of agents and due to the required properties of f_{cond} , there exists a finite time $T_{f_j} \geq 0$, when either the step (IV) applies or when one of the agents $j \in \mathcal{N}_a$ necessarily becomes passive by the function $\Pr(\cdot)$ in step (II) and remains passive till the end of the round. In the former case $T_{\odot_m} = T_{f_j}$, i.e., we directly obtain that the m -th round is finite. In the latter case, by inductive reasoning we obtain that there exists a finite time instant T_f , such that step (IV) applies, i.e., such that $T_{\odot_m} = T_f$. Again, we have that m -th round is finite.

Inductively, let $m > 1$, $t = T_{\odot_{m-1}}$, and $\Upsilon_i[j] = 0$, for all $i, j \in \mathcal{N}$ by step (IV). Using analogous arguments as above, we obtain the existence of a finite T_{\odot_m} . \square

THEOREM 11. By following the protocol above, it is guaranteed that $\forall i \in \mathcal{N}$, φ_i is satisfied by $\mathbf{x}_i(T)$ and $\|x_i(t) - x_j(t)\| < r$, $\forall (i, j) \in E(0)$ and $\forall t > 0$, where $T \rightarrow \infty$.

PROOF. The satisfaction of φ_i follows directly from the correctness of each agent's discrete plan and the fact that each round is finite by Lemma 10. At last, the distance constraints between neighbouring agents are always maintained as shown in Theorem 8. \square

5. SIMULATION

In the following case study, we simulate a team of four autonomous robots $\mathcal{N} = \{\mathfrak{R}_1, \dots, \mathfrak{R}_4\}$ subject to the dynamics (1) in a bounded, obstacle-free workspace of 40×40 meters (m). Each robot \mathfrak{R}_i is given a local sc-LTL or LTL task φ_i . All algorithms and modules were implemented in Python 2.7. Simulations were carried out on a desktop computer (3.06 GHz Duo CPU and 8GB of RAM) with a simulation stepsize set to 1ms.

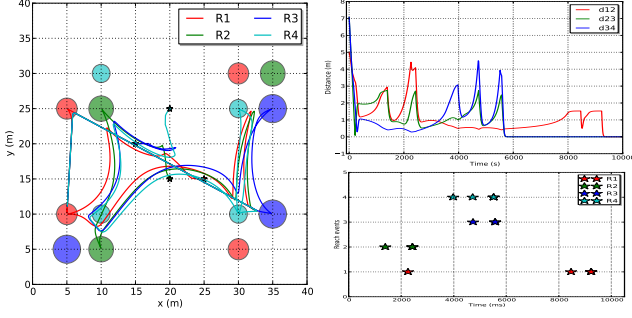


Figure 1: Left: agents' respective regions of interest in red, green, blue and cyan respectively and their trajectories after execution of switching policy from Sec. 4.3.1. All agents accomplish their sc-LTL tasks after 9s. Top-right: the evolution of pair-wise distances $\|x_{12}\|, \|x_{23}\|, \|x_{34}\|$, which all stay below $7.5m$. Bottom-right: the time instants when the agents reach their goal regions.

As shown Fig. 1, several sphere regions of interest for each agent are placed in top-left, top-right, bottom-right, and bottom-left corners of the workspace and they all satisfy Assump. 1 with $c_{\max} = 40$ and $r_{\min} = 2$:

- $\Pi_1 = \{\varpi_{1t1}, \varpi_{1tr}, \varpi_{1br}, \varpi_{1b1}\}$ shown in red;
- $\Pi_2 = \{\varpi_{2t1}, \varpi_{2tr}, \varpi_{2b1}\}$ shown in green;
- $\Pi_3 = \{\varpi_{3tr}, \varpi_{3br}, \varpi_{3b1}\}$ shown in blue;
- $\Pi_4 = \{\varpi_{4t1}, \varpi_{4tr}, \varpi_{4br}, \varpi_{4b1}\}$ shown in cyan.

The respective sets of atomic propositions (services) are $\Sigma_1 = \{\sigma_{11}, \sigma_{12}\}$; $\Sigma_2 = \{\sigma_{21}, \sigma_{22}, \sigma_{23}\}$; $\Sigma_3 = \{\sigma_{31}, \sigma_{32}, \sigma_{33}\}$; and $\Sigma_4 = \{\sigma_{41}, \sigma_{42}\}$. The regions are labeled as follows: $L_1(\varpi_{1t1}) = L_1(\varpi_{1br}) = \{\sigma_{11}\}$, $L_1(\varpi_{1tr}) = L_1(\varpi_{1b1}) = \{\sigma_{12}\}$; $L_2(\varpi_{2t1}) = \{\sigma_{21}\}$, $L_2(\varpi_{2tr}) = \{\sigma_{22}\}$, $L_2(\varpi_{2b1}) = \{\sigma_{23}\}$; $L_3(\varpi_{3tr}) = \{\sigma_{31}\}$, $L_3(\varpi_{3br}) = \{\sigma_{32}\}$, $L_3(\varpi_{3b1}) = \{\sigma_{33}\}$; and finally $L_4(\varpi_{4t1}) = L_4(\varpi_{4tr}) = \{\sigma_{41}\}$, $L_4(\varpi_{4br}) = L_4(\varpi_{4b1}) = \{\sigma_{42}\}$. The agents start from $[25, 15]$, $[20, 15]$, $[15, 20]$, and $[20, 25]$, respectively. The uniform neighboring radius is $r = 8m$ and the design parameter needed in Def. 1 is $\delta = 0.5m$. The edge set of $G(0)$ is hence $E(0) = \{(\mathfrak{R}_1, \mathfrak{R}_2), (\mathfrak{R}_2, \mathfrak{R}_3), (\mathfrak{R}_3, \mathfrak{R}_4)\}$. The upper bound by (35) is $\varepsilon < \varepsilon_{\min} \approx 0.031$ and we choose $\varepsilon = 0.03$.

We consider two cases of the agent task specifications: one with sc-LTL formulas and one with general LTL formulas.

sc-LTL Task Specifications. The local task of agent \mathfrak{R}_1 to provide service σ_{12} , then σ_{11} and at last again σ_{12} . The corresponding LTL formula is $\varphi_1 = \Diamond(\sigma_{12} \wedge \Diamond(\sigma_{11} \wedge \Diamond\sigma_{12}))$. Agent \mathfrak{R}_2 is asked to provide service σ_{21} or σ_{22} and service σ_{23} in any order, formalized as $\varphi_2^s = \Diamond(\sigma_{21} \vee \sigma_{22}) \wedge \Diamond\sigma_{23}$. The task of agent \mathfrak{R}_3 is to provide service σ_{31} or σ_{32} and service σ_{33} in any order, formalized as $\varphi_3^s = \Diamond(\sigma_{31} \vee \sigma_{32}) \wedge \Diamond\sigma_{33}$. Finally, agent \mathfrak{R}_4 is required to provide service σ_{42} , then σ_{41} and at last again service σ_{42} , represented by the LTL formula $\varphi_4 = \Diamond(\sigma_{42} \wedge \Diamond(\sigma_{41} \wedge \Diamond\sigma_{42}))$.

The synthesized discrete plans are as follows:

- $\tau_1 = (\varpi_{1b1}, \{\sigma_{12}\})(\varpi_{1t1}, \{\sigma_{11}\})(\varpi_{1b1}, \{\sigma_{12}\})$
- $\tau_2 = (\varpi_{2t1}, \{\sigma_{21}\})(\varpi_{2b1}, \{\sigma_{23}\})$
- $\tau_3 = (\varpi_{3tr}, \{\sigma_{31}\})(\varpi_{3br}, \{\sigma_{33}\})$
- $\tau_4 = (\varpi_{4br}, \{\sigma_{41}\})(\varpi_{4tr}, \{\sigma_{42}\})(\varpi_{4t1}, \{\sigma_{41}\})$

At $t = 0$, the switching policy from Sec. 4.3.1 is applied. The agent trajectories are shown in Fig. 1, where the distances between the neighboring agents along with times of reaching the agents' progressive goal regions are plotted, too.

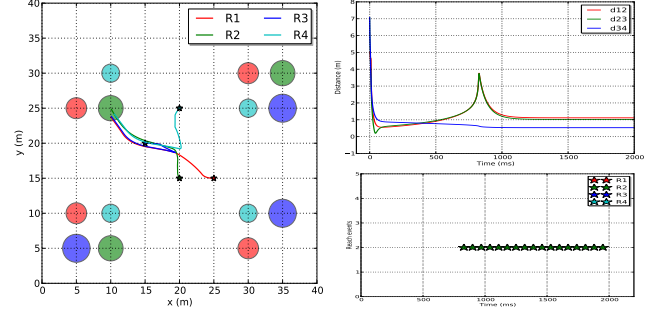


Figure 2: The system with general LTL formulas after application of switching policy from Sec. 4.3.1. All agent converge to ϖ_{2t1} and stop there due to the fact that \mathfrak{R}_2 remains active with an infinite plan to stay at ϖ_{2t1} .

General LTL task specifications. The task of agent \mathfrak{R}_1 to periodically provide both services σ_{11} and σ_{12} , represented by $\phi_1 = \Box\Diamond\sigma_{11} \wedge \Box\Diamond\sigma_{12}$. The task of agent \mathfrak{R}_2 is to periodically provide one of the services σ_{21} or σ_{22} or σ_{23} , formalized as $\phi_2 = \Box\Diamond(\sigma_{21} \vee \sigma_{22} \vee \sigma_{23})$. Finally, the tasks of agents \mathfrak{R}_3 and \mathfrak{R}_4 are $\phi_3 = \Box\Diamond(\sigma_{31} \vee \sigma_{32} \vee \sigma_{33})$, and $\phi_4 = \Box\Diamond\sigma_{41} \wedge \Box\Diamond\sigma_{42}$. The synthesized discrete plans are as follows:

- $\tau_1 = ((\varpi_{1b1}, \{\sigma_{12}\})(\varpi_{1t1}, \{\sigma_{11}\}))^\omega$
- $\tau_2 = (\varpi_{2t1}, \{\sigma_{21}\})^\omega$
- $\tau_3 = (\varpi_{3b1}, \{\sigma_{33}\})^\omega$
- $\tau_4 = ((\varpi_{4br}, \{\sigma_{41}\})(\varpi_{4tr}, \{\sigma_{42}\}))^\omega$

First, we simulated the scenario where we applied the activity switching protocol for sc-LTL formulas proposed in Sec. 4.3.1. Fig. 2 shows that the first progressive goal visited is ϖ_{2t1} . Since the agent \mathfrak{R}_2 stays active by the protocol, and its next progressive goal region is again ϖ_{2t1} , the whole system has reached its stable local minimum. Hence, all agents converge very close to one point and stop. In contrast, the activity switching protocol from Sec. 4.3.2 avoids such an unwanted behavior.

The simulation results for the activity switching protocol from Sec. 4.3.2 are illustrated in Fig. 3. The functions f_{prob} and f_{cond} were chosen in a way that allows to partially trade fairness of activity switching for increased efficiency of plan executions measured in terms of the distance traveled between consecutive visits to progressive goal regions. More specifically, an agent is not switched to passive immediately after it reaches one of its goal region. Rather than that, it has the following probability of remaining active:

$$\Pr(b_i = 1) = \begin{cases} e^{-\alpha_i \Upsilon_i[i](t - \chi_i)}, & \text{if } \Upsilon_i[i] \cdot (t - \chi_i) < \bar{\chi}_i, \\ 0, & \text{if } \Upsilon_i[i] \cdot (t - \chi_i) \geq \bar{\chi}_i, \end{cases}$$

where $\bar{\chi}_i = 5$, and $\alpha_i = 1$. The probability of remaining active decreases with the increasing time elapsed since the current round started and with the increasing number agent \mathfrak{R}_i 's own progressive goal region was visited. Note that there exists a finite $T \in (T_{\cup_{m-1}}, T_{\cup_m})$, such that $\Upsilon_i[i] \cdot (t - \chi_i) \geq \bar{\chi}_i$ for all $t \in [T, T_{\cup_m}]$, hence each agent \mathfrak{R}_i is guaranteed to be switched to passive control mode eventually.

The selected function does not necessarily yield a monotonic decrease of the total number of active agents in the team and is particularly useful when one agent has a set of goal regions whose locations are close.

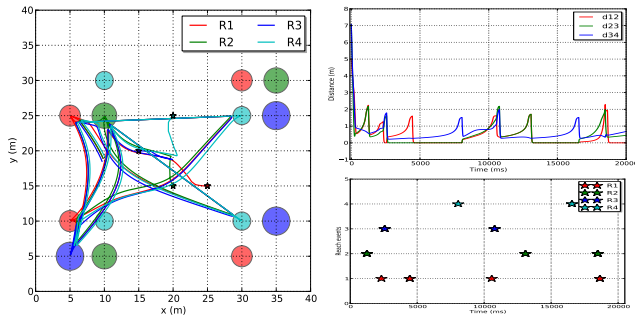


Figure 3: Left: agents' respective regions of interest in red, green, blue and cyan respectively and their trajectories after execution of switching policy from Sec. 4.3.2 for 20s. Top-right: the evolution of pair-wise distances $\|x_{12}\|$, $\|x_{23}\|$, $\|x_{34}\|$, which all stay below 7.5m. Bottom-right: the time instants when the agents reach their goal regions according to their plan.

6. CONCLUSION AND FUTURE WORK

We proposed a distributed communication-free hybrid control scheme for multi-agent systems to fulfil locally-assigned tasks as general or sc-LTL formulas, while at the same time subject to relative-distance constraints.

Future work plans include handling uncertainties in the relative state measurements and considering more complex agent dynamics. We also plan to relax the requirement on the completeness of the graph $G(t)$.

7. ACKNOWLEDGEMENTS

This work was supported by EU STREP RECONFIG: FP7-ICT-2011-9-600825 and the Swedish Research Council.

8. REFERENCES

- [1] C. Baier and J.-P. Katoen. *Principles of Model Checking*. MIT Press, 2008.
- [2] Y. Chen, X. C. Ding, A. Stefanescu, and C. Belta. Formal approach to the deployment of distributed robotic teams. *IEEE Transactions on Robotics*, 28(1):158–171, 2012.
- [3] D. V. Dimarogonas, S. G. Loizou, K. J. Kyriakopoulos, and M. M. Zavlanos. A feedback stabilization and collision avoidance scheme for multiple independent non-point agents. *Automatica*, 42(2):229–243, 2006.
- [4] I. Filippidis, D. V. Dimarogonas, and K. J. Kyriakopoulos. Decentralized multi-agent control from local LTL specifications. *IEEE Conference on Decision and Control (CDC)*, pages 6235–6240, 2012.
- [5] M. Franceschelli, M. B. Egerstedt, and A. Giua. Motion probes for fault detection and recovery in networked control systems. *American Control Conference (ACC)*, pages 4358–4363, 2008.
- [6] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer Graduate Texts in Mathematics, 2001.
- [7] M. Guo and D. V. Dimarogonas. Distributed plan reconfiguration via knowledge transfer in multi-agent systems under local LTL specifications. *IEEE International Conference on Robotics and Automation (ICRA)*, pages 4304–4309, 2014.
- [8] M. Guo, J. Tumova, and D. V. Dimarogonas. Cooperative decentralized multi-agent control under local LTL tasks and connectivity constraints. *IEEE Conference on Decision and Control (CDC)*, 2014. To appear.
- [9] M. Guo, M. M. Zavlanos, and D. V. Dimarogonas. Controlling the relative agent motion in multi-agent formation stabilization. *IEEE Transactions on Automatic Control*, 59(3):820–826, 2014.
- [10] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge university press, 2012.
- [11] M. Ji and M. B. Egerstedt. Distributed coordination control of multi-agent systems while preserving connectedness. *IEEE Transactions on Robotics*, 23(4):693–703, 2007.
- [12] H. K. Khalil and J. W. Grizzle. *Nonlinear systems*. Prentice Hall, 2002.
- [13] M. Kloetzer, X. C. Ding, and C. Belta. Multi-robot deployment from LTL specifications with reduced communication. *IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, pages 4867–4872, 2011.
- [14] O. Kupferman and M. Y. Vardi. Model checking of safety properties. *Formal Methods in System Design*, 19:291–314, 2001.
- [15] S. G. Loizou and K. J. Kyriakopoulos. Automated planning of motion tasks for multi-robot systems. *IEEE Conference on Decision and Control (CDC)*, pages 78–83, 2005.
- [16] J. Lygeros, F. Ramponi, and C. Wiltsche. Synthesis of an asynchronous communication protocol for search and rescue robots. *European Control Conference (ECC)*, pages 1256–1261, 2013.
- [17] M. Mazo and P. Tabuada. Decentralized event-triggered control over wireless sensor/actuator networks. *IEEE Transactions on Automatic Control*, 56(10):2456–2461, 2011.
- [18] W. Ren, R. W. Beard, and E. M. Atkins. A survey of consensus problems in multi-agent coordination. *American Control Conference (ACC)*, pages 1859–1864, 2005.
- [19] E. Rimon and D. E. Koditschek. Exact robot navigation using cost functions: the case of distinct spherical boundaries in E^n . *IEEE International Conference on Robotics and Automation (ICRA)*, pages 1791–1796, 1988.
- [20] I. Saha, R. Ramaithitima, V. Kumar, G. Pappas, and S. Seshia. Automated composition of motion primitives for multi-robot systems from safe LTL specifications. *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, 2014.
- [21] J. Tumova and D. V. Dimarogonas. A receding horizon approach to multi-agent planning from LTL specifications. *American Control Conference (ACC)*, pages 1775 – 1780, 2014.
- [22] A. Ulusoy, S. L. Smith, X. C. Ding, C. Belta, and D. Rus. Optimality and robustness in multi-robot path planning with temporal logic constraints. *International Journal of Robotics Research*, 32(8):889–911, 2013.
- [23] M. M. Zavlanos, M. B. Egerstedt, and G. J. Pappas. Graph-theoretic connectivity control of mobile robot networks. *Proceedings of the IEEE*, 99(9):1525–1540, 2011.